International Journal of Theoretical Physics, Vol. 38, No. 1, 1999

Irreversibility, Lax–Phillips Approach to Resonance Scattering and Spectral Analysis of Non-Self-Adjoint Operators in Hilbert Space

B. Pavlov

Received August 7, 1998

Since the publication of the very first papers on quantum mechanics the theory of self-adjoint operators in Hilbert space has been a basic tool of quantum theories. It turns out that the description of the irreversible dynamics of complex systems requires the development of the spectral theory of non-self-adjoint operators as well. In this paper we consider the Hilbert space version of the theory of dissipative operators, which appear as generators of the evolution reduced to a properly selected observation subspace. The spectral analysis of these operators is based on ideas of the functional model and dilation theory rather than on traditional resolvent analysis and Riesz integrals. The role of the parameter of the functional model is played by an analytic function—the characteristic function—which is interpreted and calculated as a scattering matrix for the relevant scattering problem. Thus the most important object of the spectral analysis of dissipative operators appears as an element of spectral analysis of a self-adjoint spectral problem. This paper is intended both as an introduction and a sort of bilingual text for specialists in harmonic analysis and operator theory who are interested in mathematical problems of the description of irreversible dynamics. The last part describes original results of the author published in different journals during the last decade.

1. SPECTRAL ANALYSIS OF SELF-ADJOINT OPERATORS AND QUANTUM LOSCHMIDT PARADOX

The first really serious success of the spectral theory of operators was achieved actually almost 200 years ago by the French bureaucrat Jean Baptist Fourier, appointed by Napoleon a prefect of the department of Val d'Isere. In 1822 Fourier published *Theorie Analytique de la Chaleur* [1], where the heat exchange in a rectangular solid block Ω was described under the assump-

[1] Department of Mathematics, University of Auckland, Auckland, New Zeland, and International Solvay Institute for Physics, Brussels, Belgium.

0020-7748/99/0100-02 1$16.00/0

tion that the temperature at the surface $\partial\Omega$ was given. Fourier reduced this problem to a partial differential equation ("heat equation") for the corresponding temperature field $T(x, y, z, t)$:

$$\frac{\partial T}{\partial t} - \kappa^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = 0$$

with a positive coefficient κ^2 for heat conductivity and the proper initial and boundary conditions:

$$T(x, y, z, 0) = T_0(x, y, z), \qquad (x, y, x) \in \Omega$$

$$T(x, y, z, t) = 0, \qquad (x, y, z) \in \partial\Omega, \quad t \geq 0$$

Fourier suggested a remarkable method—the *Fourier method*—for the solution of this problem. Having found all "spatial modes"—*the eigenfunctions*—of the Laplace operator in the domain Ω, i.e., all nonzero solutions φ_ω to the stationary homogeneous equation

$$-\kappa^2 \left[\frac{\partial^2 \varphi_\omega}{\partial x^2} + \frac{\partial^2 \varphi_\omega}{\partial y^2} + \frac{\partial^2 \varphi_\omega}{\partial z^2} \right] = \lambda_\omega \varphi_\omega$$

he suggested to use for the solution of the initial nonstationary problem an anzatz represented in a form of a series over all spatial modes φ_ω with time-dependent coefficients. Inserting this ansatz into the heat equation, he found that the coefficients are just proportional to the exponentials of the eigenvalues λ_ω:

$$T = \sum_\omega e^{-\lambda_\omega t} u_\omega \varphi_\omega$$

Then he defined the amplitudes u_ω from the initial condition having represented it in form of the *Fourier series:*

$$\mathrm{T}_0 = \sum_\omega u_\omega \varphi_\omega$$

The convergence of this series follows from the *orthogonality* of spatial modes

$$\int \varphi_\omega \bar{\varphi}_{\omega'} \, dm = \delta_{\omega,\omega''}$$

and an infinite-dimensional version of the Pythagorean theorem ("Parseval identity")

$$\langle T_0, T_0 \rangle = \sum_\omega |u_\omega|^2$$

An elegant geometric interpretation of the Fourier method was found almost 100 years later by David Hilbert, and the infinite-dimensional function space with the dot-product similar to one defined above by the integral is now one of the most popular function spaces, and bears Hilbert's name [2].

But the real celebration of spectral analysis coincides with the beginning of quantum physics. The needs of physics stimulated the development of the spectral theory of so-called Schrödinger operators in this century.

The simplest quantum Hamiltonian—Schrödinger operator—in proper time-space scale looks like a sum of the Laplacian and a multiplication operator by some real continuous function q—the "potential":

$$Lu = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + q(x, y, z)u \equiv -\Delta u + q(x, y, z)u$$

The state of a quantum particle is defined by the wave function, which satisfies a nonstationary Schrödinger equation similar to the heat equation, but a complex one:

$$\frac{1}{i}\frac{\partial \psi}{\partial t} = L\psi$$

$$\psi\big|_{t=0} = \psi_0$$

and this can be constructed by the Fourier method, provided the eigenfunctions of the corresponding Schrödinger operator L are found:

$$L\varphi_\omega = \lambda_\omega \varphi_\omega$$

The solution of the nonstationary Schrödinger equation can be represented as the action of the *dynamical group* V_t on the initial wave vector:

$$\psi(t) = \sum_\omega e^{i\lambda_\omega t}\varphi_\omega \psi_\omega^0$$

$$= e^{iLt}\psi_0 \equiv V_t\psi_0$$

If the potential is real ($q = \overline{q}$) and semibounded from below ($\inf q > -\infty$), then the Hamiltonian L is "self-adjoint"; in particular its quadratic form is real:

$$\overline{\langle L\psi, \psi\rangle} = \langle \psi, L\psi\rangle$$

and the corresponding eigenfunctions—"pure quantum states"—are orthogonal in the proper (or improper) sense and can be normalized

$$\langle \varphi_\omega, \varphi_{\omega'}\rangle = \delta_{\omega\omega'}$$

so the coefficients ψ_ω^0 of the solution of the nonstationary equation can be defined from the initial conditions as orthogonal projections: $\psi_\omega^0 = \langle \varphi_\omega, \psi_{(0)} \rangle$. The "mean value" of any other *quantum observable* which is represented by a self-adjoint operator A in the Hilbert space of wave functions is defined in the fixed state ψ by the quadratic form of this operator with given quantum state

$$A(\psi) = \langle \psi, A\psi \rangle$$

For instance, the value of the quantum mechanical momentum operator P_x

$$P_x = i \frac{d}{dx}$$

is given by the integral over the corresponding *configuration space*:

$$P_x(\psi) = \int_{R_3} i \frac{d\psi}{dx} \overline{\psi}\, dm$$

The set of all eigenvalues—*the discrete spectrum*—of the Hamiltonian is one the most important characteristics of a quantum system, since it gives the list of all possible values of energy λ_ω which the quantum system can acquire,

$$L(\varphi_\omega) = \langle \varphi_\omega, L\varphi_\omega \rangle = \lambda_\omega$$

Generally the spectrum $\{\lambda_\omega\} \equiv \sigma(L)$ of the Hamilton L can contain both discrete and continuous part if the system is not bounded in space or if the potential has strong singularities.

The investigation of the structure of the spectrum, in particular, the distribution of eigenvalues, the convergence of eigenfunction expansions in the form of Fourier series or integrals, and the asymptotic behavior of the solutions of stationary and nonstationary equations on a large time and space scale are the main problems of the spectral analysis of self-adjoint differential operators, in particular the Schrödinger operators. This branch of the theory of operators and mathematical physics has been intensely developed during the past few decades [21], aside from some important details which have put essential physical restrictions on our considerations.

Notice first that the most straightforward results concerning the convergence of eigenfunction expansions of the solution of the Schödinger equation follow from the orthogonality of the eigenfunctions via the Parseval identity. This is equivalent to the conversation law for the wave function:

$$\langle \psi(t), \psi(t) \rangle = \langle \psi(0), \psi(0) \rangle = \sum \psi_\omega^0 \overline{\psi_\omega^0}$$

Interpreting the square modulo of the wave function $\psi(x)\overline{\psi(x)}$ as a probability density to find the particle at the point x, we can assign an important physical meaning to the mentioned conservation law: it means, that the total probability to find the particle in the configuration space R_3 is equal to 1 and does not change in course of evolution governed by the *self-adjoint* Hamilontian L. In other words, the dynamical group corresponding to the Schrödinger equation is *unitary* in this case,

$$\langle V_t\psi_0, V_t\psi_0\rangle = \langle\psi_0, \psi_0\rangle = \int_{R_3} \left|\psi_0\right|^2 dm$$

This elegant statement actually shows that the evolution of the wave function described by the above equation is *reversible*.

This fact becomes even more obvious if we consider the quantum evolution in the Hilbert space of *density matrices* $\rho(x, y)$, which is just a tensor product of two copies of the space of wave functions. Density matrices can be interpreted as kernels of Hilbert–Schmidt integral operators acting in the space of wave functions. Denoting these operators by ρ, we see that the evolution of the simplest density matrix $\rho(x, y) = \psi(x) \times \psi'(y)$ is given by the formula

$$\rho(x, y; t) = e^{iLT}\psi(x) \times \overline{e^{iLt}\psi'(y)}$$

and thus it is governed by the operator which is connected to the corresponding Hamiltonian in a very simple way:

$$L^*\rho = L\rho - \rho L \equiv [L, \rho]$$

This operator L^* is called *Liouvillian*; it has a symmetric spectrum [3],[2] formed of all possible differences: $\{\lambda_{\omega\omega'} = \lambda_\omega - \lambda_{\omega'}\}$, and the eigenfunctions of it are just direct products of the corresponding eigenfunctions of the Hamiltonian, $\rho_{\omega\omega'}(x, y) = \varphi_\omega(x)\overline{\varphi_{\omega'}}(y)$,

$$L^*\rho_{\omega\omega'} = \lambda_{\omega\omega'}\rho_{\omega\omega'}$$

The whole picture of quantum physics can be written in terms of density matrices and Liouvillians. In a way it is more natural than quantum mechanics in terms of wave functions, since actually physicists measure not the values of the energy λ_ω, but the *transition energies* $\lambda_{\omega\omega'} = \lambda_\omega - \lambda'_\omega$, which are exactly eigenvalues of the corresponding Liouvillians. Moreover, considering Liouvillians instead of Hamiltonians, we avoid some trival quantum divergences, such as the infinite energy of the vaccum state of an infinite system of quantum oscillators.

[2]The paper by Sophn [3] contains an error considering the singular part of the spectrum.

Writing down the solution of the nonstationary equation for the Liouvillian for some conservative quantum system (for instance, the universe),

$$\frac{1}{i}\frac{d\rho}{dt} = L^*\rho \equiv [L, \rho]$$

we can change (at least "in principle") the initial conditions at some moment of time in such a way that the Fourier coefficients which correspond to the opposite eigenvalues just change places. Then up to the trivial isomorphism $\rho_\lambda \to \rho_{-\lambda}$, the evolution of the system will go in the opposite direction. Thus, in principle, we can invert time. The described paradox is an analog of the celebrated Loschmidt paradox concerning the time reversibility in Newtonian mechanics. That paradox was suggested to disprove the Boltzmann's H-theorem on approaching equilibrium in a *large system* of Newtonian particles. But in fact the Loschmidt paradox just showed that standard Newtonian mechanics cannot describe the approach to equilibrium. Similarly our quantum version of the Loschmidt paradox shows that standard quantum mechanics based on the spectral theory of *Self-adjoint operators* meets some difficulties when describing the dynamics of large quantum systems.

2. IRREVERSIBLE DYNAMICS AND THE MINIMAL DILATION

It is important that the above philosophical paradox is relevant to any extended (not necessarily large) quantum or classical system, since we never can invert time just because we never can collect the complete set of initial data for any extended system. This straightforward explanation of irreversibility, carefully analyzed by D. Ruelle, is not complete [19]. The Brussels school connects the phenomenon of irreversibility of dynamics with intrinsic properties of the system, which are revealed by proper *rigging* of the corresponding Hilbert space [8]. The high price they pay for it is the disappearance of Hilbert structure. The general spectral theory of non-self-adjoint operators in rigged spaces is not yet developed, though there is a series of elegant soluble models. That is why in this paper we keep a simpler point of view, assuming that the irreversibility is caused by the reduction of the dynamics onto the properly chosen observation subspace. This means that we are dealing now not with the whole dynamical group V_t satisfying certain conservation laws, but with the *compression* of this group onto the proper observation subspace K,

$$P_K V_t\big|_K$$

A simple but typical example of an acoustic system for which the compressed evolution reveals the semigroup properties is given by a resonator with a small opening, or by the inside of a dome with an open door connecting it to the surrounding space. In this case the restriction of the wave dynamics onto the inner part of the system gives to our surprise a *reduced dynamics* as well, since the "inside" evolution is defined by the "inside" data only and the emitted waves never return [7]. This reduced dynamics is described by a semigroup generated by some *non-self-adjoint* ("dissipative") operator B:

$$K_K V_t \big|_K = e^{iBt}$$

which has generally a nonorthogonal system of eigenfunctions and a complex (nonreal) spectrum.

The elegant picture of spectral analysis described above for self-adjoint operators fails in this case, since neither the Parseval identity nor the conservation law is available for reduced dynamics and for eigenfunctions of the corresponding generator. But in some way the picture of the evolution becomes more realistic, since the unpleasant paradox of existence of (though conceivable!) inverse evolution is excluded, since an essential piece of information is lost in course of evolution, for instance, all the components of the wave process which correspond to the complex eigenvalues λ, $\Im\lambda > 0$, go to zero when $t \to \infty$.

Thus we see that the investigation of realistic situations includes—at least sometimes—an essential *reduction* of the space of all initial data. This means that the spectral analysis of self-adjoint operators in these cases must be replaced by the spectral analysis of operator functions which we get by reducing the unitary dynamical group onto proper observation subspaces.

$$P_K V_t \big|_K = -\frac{1}{2\pi i} \oint P_K R_\lambda \big|_K e^{i\lambda t}\, d\lambda$$

or maybe even to the spectral analysis of some non-self-adjoint operators. But at the same time it is convenient for technical reasons to retain the main advantages of the self-adjoint theory such as orthogonality of eigenfunctions and the Parseval identity, to guarantee the convergence of the Fourier series representing the reduced solutions of the corresponding nonstationary equations.

Fortunately, both these controversial demands can be satisfied.

Let us consider the general *Lax–Phillips case*, when the reduced dynamics is represented by some evolution semigroup. This takes place if the unitary dynamical group generated by our Hamiltonian or Liouvillian possesses an orthogonal pair of incoming and outgoing subspaces $D_\pm$:

$$e^{iLt} D_+ \subset D_+, \quad t > 0, \qquad e^{iLt} D_- \subset D_-, \quad t < 0$$

$$\langle u_+, u_- \rangle = 0, \qquad u_\pm \in D_\pm$$

The orthogonal complement K of $E_+ \oplus D_-$, which is a *coinvariant subspace* of the forward evolution reduced onto $H \ominus D_-$, can be chosen as an observation subspace. In the Lax–Phillips case the original unitary dynamical group compressed onto this subspace proves to be a *contracting* dynamical semigroup and thus is generated by some dissipative generator B. The resolvent of this operator in the lower half-plane coincides with the compressed resolvent of the generator of the original unitary dyamical group.

To observe these facts, we need a small portion of the geometry of Hilbert spaces. Let us check the first statement, following ref. 7. Using the orthogonal decomposition $H' = D_- \oplus K \oplus D_+$, we transform the expression for the reduced dynamics in the following way:

$$\begin{aligned} P_K e^{iL(t_1+t_2)} P_K &= P_K e^{iLt_1}\{P_{D+} + P_K + P_{D-}\} e^{iLt_2} P_K \\ &= P_K e^{iLt_1} P_{D_+} e^{iLt_2} P_K + P_K e^{iLt_1} P_K e^{iLt_2} P_K \\ &\quad + P_K e^{iLt_1} P_{D_-} e^{iLt_2} P_K \end{aligned}$$

The first and third terms in the right side vanish due to the invariance of $D_\pm$ if $t_1, t_2 \geq 0$ and the central term is represented in the form

$$\begin{aligned} P_K^{iL(t_1+t_2)} P_K &= P_K e^{iLt_1} P_K P_K e^{iLt_2} P_K \\ &= P_K e^{iLt_1} P_K \times P_K e^{iLt_2} P_K, \quad t_1, t_2 \geq 0 \end{aligned}$$

Thus the semigroup property is fulfilled. Since the projections P_K are bounded operators and the unitary dynamical group is strongly continuous, then the reduced semigroup is a strongly continuous contracting semigroup as well, hence it has a dissipative generator B,

$$P_K e^{iLt} P_K = e^{iBt}, \qquad \Im B \geq 0$$

which means that the conservation law is replaced by the dissipation.

Let us check that the resolvent of the dissipative generator B is connected by a simple formula to the resolvent of the generator L of the unitary dynamics. In fact, integrating the last equation multiplied by $e^{-i\lambda t}$ over t on the interval $[0, \infty)$, we get immediately for $\Im \lambda < 0$

$$\begin{aligned} P_K \frac{I}{L - \lambda I} P_K &= -\frac{1}{i} \int_0^\infty P_K e^{i(L-\lambda)t} P_K \, dt \\ &= -\frac{1}{i} \int_0^\infty e^{i(B-\lambda)t} \, dt = \frac{I}{B - \lambda I} \end{aligned}$$

The last formula means that the resolvent of the initial self-adjoint operator framed by projections onto the observation subspace possesses an analytic continuation onto the lower half-plane as the resolvent of a certain dissipative operator and thus it can be continued also onto the upper half-plane as well with singularities at the spectrum of this dissipative operator.

Unfortunately the described elegant representation of the analytical continuation of the reduced resolvent $P_K[I/(L - \lambda I)]P_K$ is possible only for operators which serve as generators of unitary groups possessing Lax–Phillips properties. In particular, the operator L must contain a part which is unitarily equivalent to the quantum mechanical momentum operator $(1/i)\, d/dx$ and thus has an absolutely continuous spectrum covering the whole real axis with constant multiplicity. Expanding the Lax–Phillips techniques beyond this class requires development of harmonic analysis on multiconnected domains [16, 18].

The Lax–Phillips approach is a powerful tool for the investigation of the spectral properties of generators of the reduced dynamics. In particular we get the *functional model* of the dissipative operator, writing it down in the spectral representation of the original self-adjoint generator of the unitary evolution group. Actually different spectral representations give different types of functional models [10]. But generally these functional models permit us to reduce important geometrical questions of spectral theory to equivalent questions of the theory of analytic functions (see Section 3).

Conversely, consider some dissipative generator B of a contracting semigroup. To build the functional model of it we need some *restoring construction* which could restore the complete unitary dynamical group from the reduced one. Let us consider a simple but representative example. We know from our everyday experience that the acoustic field inside a dome is not very much affected by events outside. This means that possibly we make only a small error when replacing the "outdoor" part of the whole Hamiltonian by some minimal artifical model construction, to accomplish the contracting dynamics inside and create a new *minimal* unitary (with proper conservation laws fulfilled) dynamical group V'_t acting on some new Hilbert space $H' \supset K$, such that the *compression* of V'_t onto the coinvariant subspace K gives the same result as the compression of the original dynamics V_t on it:

$$P_K V_t\big|_K = e^{iBt} = P_K V'_t\big|_K, \qquad t > 0$$

This new unitary dynamical group is called the *minimal unitary dilation* of the contracting evolution semigroup e^{iBt}. One can show that the minimal unitary dilation is uniquely defined up to some trivial unitary isomorphism. In particular this means that the minimal unitary dilation is unitarily equivalent to the proper part of the complete dynamical group V_t reduced onto the invariant subspace H_K generated by the observation subspace that is the coinvariant subspace K:

$$H_K = \bigvee_{t=-\infty}^{+\infty} V_t K$$

Thus the construction of the unitary dilation, by attaching ''artificial'' channels, results in separating *the part of the real dynamical group* acting in the minimal subspace of the whole Hilbert space which proves to be connected to K in the course of evolution. To construct the minimal self-adjoint dilation it is sufficient to accomplish K by two orthogonal channels—incoming and outgoing ones $D_\pm = L_2(0, \pm\infty; E)$—with simple *shifts* acting on them [13].

The *shift operator* is an important detail of the proposed construction. We discuss now some typical features of this important object. Consider first the simplest example of the *shift group* in $L_2(R)$:

$$V_t\colon \quad u(x) \to u(x - t)$$

It is obvious that this is a unitary commutative group since

$$\int_0^\infty |u(x)|^2\, dm = \int_0^\infty |u(x - t)|^2\, dm, \qquad V_t V_s u = V_{t+s} u$$

and the subspaces $D_\pm = L_2(R_\pm)$ are obviously invariant subspaces of it:

$$V_t D_+ \subset D_+, \quad t > 0, \qquad V_t D_- \subset D_-, \quad t < 0$$

They are obviously orthogonal in the sense of Hilbert space. The spectral representation of the shift group is given by F Fourier which can be interpreted as a spectral decomposition on improper eigenfunctions $\varphi_k = e^{-ikx}$ of the generator of the group, the quantum mechanical momentum operator P:

$$P = i\frac{d}{dx}, \qquad P\varphi_k = k\varphi_k$$

$$F\colon \quad u \to \langle \varphi_k, u\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx} u(x)\, dm$$

F transforms $D_\pm$ isometrically into the corresponding Hardy classes $H^2_\pm$ of square-integrable functions, which possess an analytic continuation onto the upper or the lower half-planes of the variable k, respectively. The orthogonal projection onto H^2_+ in $L_2(R)$ is given by the Cauchy-type integral

$$[P_+ f](x) = \frac{1}{2\pi i} \int_R \frac{f(s)}{s - (x + i0)}\, ds$$

The components of the dilations of dissipative operators inside artificial incoming and outgoing channels attached to the initial subspace K are usually constructed of properly reduced parts of the momentum operator. As long ago as 1946 A Beurling described all invariant subspaces of shift operator in the scalar case. The general case was investigated H. Helson, V. Potapov, and B. Sz-Nagy and C. Foias; see ref. 6 for details and for a complete bibliography.

The problem of description of all invariant subsapces of a shift group looks more natural in the spectral representation of the shift group (Fourier representation), where the shift is replaced by the multiplication

$$F\colon \quad V_t \to e^{ikt} *$$

Actually in the Fourier representation the invariant subspaces of right shifts $t > 0$ contained in H^2_+ are parametrized by the *common zero-sets* $\sigma(D_+)$ of functions from D_+:

$$\sigma(D_+) = \{z\colon F(z) = 0 \quad \text{for each } f\colon f \in FD_+ \subset H^2_+\}$$

which actually serve as maximal ideals of the algebra of all bounded analytic functions in upper half-plane. For any zero-set σ one can construct an associated *inner function* S_σ; $S_\sigma(\lambda) = 0$, $\lambda \in \sigma$, which is a contracting analytic function in the upper half-plane. If, for instance (in the scalar case)

$$\sigma = \{k_l\}, \quad l = 1, 2, 3, \ldots; \quad \Im k_l > 0$$

is the set of common zeros of all functions from D_+, then the corresponding function is represented in the form of a *Blaschke product* which is convergent in the upper half-plane provided the argument of each factor is properly renormalized:

$$b(k) = \prod \frac{k - k_l}{k - \bar{k}_l} e^{-i\eta_l}$$

Here η_η is the argument of the factor $(k - k_l/k - k_l)$ at the complex point $k = i$. The common zeros of the "exponential order" at the boundary are connected with so-called *singular inner functions* parametrized by the positive measures μ on the real axis which are singular in respect to the Lebesgue measure and can have a positive "atom" μ_∞, $\mu_\infty \geq 0$, at infinity:

$$\vartheta(k) = \exp i\left\{\mu_\infty k + \int_{-\infty}^{+\infty} \frac{1 + ks}{s - k}\, d\mu(s)\right\}$$

All invariant subspaces D_+ of the shift contained in $L_2(R_+)$ are described in the spectral representation as parts of H^2_+, parametrized by general inner functions $S(k) = b(k)\vartheta(k)$:

$$D^S_+ = SH^2_+$$

The zeros of the Blashke product, the support of the measure μ, and possibly ∞ constitute the spectrum of the total inner function and play the role of common zeros of elements of the corresponding invariant subspace.

The constructed inner function S plays an important role in the description of the dynamics of the reduced system. It can be calculated in terms of the asymptotic behavior of solutions of the spatial modes of the dilated self-adjoint problem, being an object of the self-adjoint theory [9]—a *scattering matrix* for some pair of self-adjoint operators.

Let us consider the procedure of constructing the self-adjoint dilation in the most interesting case of a dissipative operator of Schrödinger type with the complex potential $q(x) = r(x) + ig^2(x)$ in $L_2(R^3)$:

$$Bu = -\Delta u + q(x)u \equiv Au + ig^2 u$$

The incoming and outgoing channels can be constructed [13] as spaces of square-integrable functions on $(-\infty, 0)$ and $(0, \infty)$, respectively, taking values in $L_2(R^3)$. The corresponding minimal self-adjoint dilation is defined on smooth elements of the orthogonal sum $H' = D_- \oplus L_2(R^3) \oplus D_+$,

$$U = \begin{pmatrix} u_-(\xi) \\ u(x) \\ u_+(\xi) \end{pmatrix}$$

satisfying the boundary condition

$$[u_-(0) - u_+(0)] = ig(x)u(x)$$

in the following way:

$$L'U = \begin{pmatrix} 2i\dfrac{d}{d\xi}u_-(\xi) \\ Au + g(x)\,[u_-(0) + u_+(0)] \\ 2i\dfrac{d}{d\xi}u_+(\xi) \end{pmatrix}$$

Using the dilation L', we can construct the corresponding unitary dynamical group $V'_t = e^{iL't}$ which proves to be the dilation of the contracting evolution semigroup e^{iBt}. The action of this new group on incoming and outgoing channels $D_\pm$ is in fact represented by shifts

$$V'_t\big|_{D_-}: \quad u_-(\xi) \to u_-(\xi - t); \qquad \xi, \xi - t < 0$$

$$V'_t\big|_{D_+}: \quad u_+(\xi) \to u_+(\xi - t); \qquad \xi, \xi - t > 0$$

These shifts bring the flow of the "global" wave function U from D_- to K and from K to D_+ (for positive t) to provide the "global" conservation law. If there are no other incoming and outgoing channels except the constructed $D_\pm$, then all the flow incoming from D_- will eventually go out into D_+, similar to the above example of the shift group in $L_2(R)$. In this case the dilated dynamical group is unitarily equivalent to the shift group, and in the proper "incoming" spectral representation the subspace D_- is represented by the Hardy class H^2_-, and D_+, being orthogonal to D_-, is represented by some invariant subspace of the shift group (e^{ikt}, $t > 0$) of Beurling's type $D_+ = SH^2_+$ parametrized by the corresponding inner function S. This inner function can be calculated in an explicit form from the asymptotics of the eigenfunctions of the dilated operator L' associated with the corresponding incoming spectral representation [10].

The (improper) eigenfunctions $\Phi = \{\varphi_-, \varphi, \varphi_+\}$ of the (absolutely) continuous spectrum of the dilated operator L' are given by the *bounded* solutions of the corresponding homogeneous equation

$$L'\Phi_\lambda = \lambda\Phi_\lambda$$

which satisfy the boundary conditions and the *causality conditions* of the analytical continuation of the component φ_- into the upper half-plane and the components φ, φ_+ into the lower half-plane. These components $\varphi_\pm$ have to coincide with the eigenfunctions of the momentum operator in the incoming channel and are proportional to the eigenfunctions of the momentum operator in the outgoing channel with some coefficient $S^+(k) = I + T^+(k)$, which is called the corresponding (adjoint) *scattering matrix*. Here T is some compact integral operator in $E = L_2$ (supp g) with the kernal $T(x, y; k)$, which is called the *scattering amplitude*:

$$\text{in } D_-: \quad \varphi_{k,y}(x, \xi) = \delta(x - y)e^{-ik\xi}$$

$$\text{in } D_+: \quad \varphi_{k,y}(x, \xi) = \delta(x - y)e^{-ik\xi} + T^+(x, y; k)e^{-ik\xi}$$

The spectral representation connected with the eigenfunctions $\Phi_{k,y}$ assigns D_- to $H^2_-(E)$ and D_+ to $SH^2_+(E)$, since for $u_\pm \in D_\pm$ we have

$$\langle \varphi_{k,x}, u_- \rangle = \int_{-\infty}^{0} e^{ik\xi} u_-(\xi, x)\, d\xi$$

$$\langle \varphi_{k,x}, u_+ \rangle = \int_{0}^{+\infty} e^{ik\xi} S(k) u_+(\xi, x)\, d\xi$$

$$= \int_{\text{supp } g} S(x, y; k) \int_{0}^{+\infty} e^{ik\xi} u_+(\xi, y)\, d\xi\, dy$$

Consider once more the case when there are no other incoming and outgoing channels except $D_\pm$ (see ref. 10 for the general case). In this case the scattering matrix S is unitary. Moreover, it follows from the orthogonality $D_\pm$ that $SH^2_+ \subset H^2_+$, i.e., S is an analytic function of the spectral variable k. Representing the subspaces D_-, K, D_+, and the operator B in terms of this incoming spectral representation U^J_-

$$U^J_- D_- \to H^2_-$$

$$U^J_- D_+ \to SH^2_+ \subset H^2_+$$

$$U^J_- K \to H^2_+ \ominus SH^2_+ \equiv K$$

$$U^J_- e^{iBt} (U^J_-)^{-1} = P_K e^{ikt}\Big|_K$$

we get the so-called *Functional model* of the dissipative operator B, which serves as a powerful tool to translate the most important questions of spectral theory into the language of the theory of analytic functions. The first variant of the functional model was constructed by Sz-Nagy and Foias [6]. The only parameter of this model is given by the scattering matrix S, which was defined independently in an acoustic problem by Lax and Phillips [7] as a *unitary* coefficient $S(k)$ connecting the "incoming" and "outgoing" spectral representations $u_\pm$.

$$U^J_-(k) = S(k) U^J_+(k)$$

with U^J_+ defined for D_+ in the same way as U^J_- is defined for D_-, that is, j_+: $D_+ \to \mathrm{H}^2_+(E)$. Later Adamjan and Arov [9] found deep connections between the Sz-Nagy–Foias functional models and the scattering theory of Lax and Phillips. In the general case the scattering matrix proves to be an analytic contracting function in the upper half-plane Im $k > 0$:

$$\left|S(k)\right| \leq 1, \qquad \text{Im } k \geq 0$$

Moreover, it coincides with the characteristic function of the dissipative operator discovered in the 1950s by Livschic [5].

3. SPECTRAL ANALYSIS OF THE NON-SELF-ADJOINT OPERATOR FROM THE POINT OF VIEW OF ITS SELF-ADJOINT DILATION

We saw that the scattering matrix appears in the spectral theory of dissipative operators as an object connected with some self-adjoint operator and can be calculated by means of the self-adjoint theory. For instance, in the case of the Schrödinger operator discussed above it was calculated as an adjoint transmission coefficient S^+ for eigenfunctions of the continuous spectrum which are "initiated" by exponentials $e^{-ik\xi}h$ in D_- and look like $e^{-ik\xi}S^+(k)h$ in D_+. It is important that the whole spectral picture of the dissipative operator B is encoded in S.

Let us consider the simplest example, assuming that the scattering matrix is a scalar inner function $S(k) = \theta(k)\Pi(k)$ in the upper half-plane $\Im k > 0$ with proper behavior at infinity: $\mu_\infty = 0$. We calculate the resolvent of the generator B of the reduced dynamics $P_K U_t|_K$ in the *incoming* spectral representation of the underlying unitary dynamics U_t assuming that $D_- = H^2_-$, $D_+ = SH^2_+$, $U_i \equiv e^{ikt}$. Obviously (see Section 2) for $\Im\lambda < 0$ we have

$$(B - \lambda I)^{-1} f = \frac{1}{k-\lambda} f, \qquad f \in K$$

We check that the expression in right side coincides with

$$\frac{f - S[S^+ f(\bar{\lambda})]}{k-\lambda}$$

For $f \in K$ we have $S^+ f \in \mathrm{H}^2_-$, hence $[S^+ f(\bar{\lambda})]$ is defined. Then $\{f - S[S^+ f(\lambda)]\}/(k-\lambda) \in H^2_+$. On the other hand, for any $g_+ \in H^2_+$ we have

$$\left\langle \frac{f - S[S^+ f(\bar{\lambda})]}{k-\lambda}, Sg_+ \right\rangle = \left\langle \frac{S^+ f - [S^+ f(\bar{\lambda})]}{k-\lambda}, g_+ \right\rangle = 0$$

since $\{S^+ f - [S^+ f(\bar{\lambda})]\}/(k-\lambda) \in H^2_-$.

Similarly for $\Im\lambda > 0$ the formula

$$(B - \lambda I)^{-1} f = \frac{f - SS^{-1}(\lambda) f(\lambda)}{k-\lambda}$$

can be checked. One can see from the last formula that the spectrum of the operator B coincides with the spectrum of the inner function S.

Another simple exercise is to check that the eigenfunction ψ_ζ of the operator B which corresponds to simple zero ζ of Π is given by the formula.

$$\psi_\zeta(k) = \sqrt{2\Im\zeta}\,\frac{\theta\Pi(k)}{k-\zeta}$$

and the eigenfunction of $\phi_{\bar\zeta}$ of the adjoint operator B which corresponds to the eigenvalue ζ is just a Cauchy kernel:

$$\phi_{\bar\zeta} = \frac{\sqrt{2\Im\zeta}}{k-\bar\zeta}$$

One can see now that the systems of eigenfunctions $\{\psi_\zeta\}$, $\{\phi_{\bar\zeta}\}$ are biorthogonal

$$\langle\psi_\zeta, \phi_{\bar\zeta'}\rangle = \delta_{\zeta,\zeta'}, \qquad \langle\psi_\zeta, \phi_{\bar\zeta}\rangle = \Im\zeta\,\frac{dS}{dk}\bigg|_\zeta$$

and, at least formally, the spectral decomposition in eigenfunctions of the discrete spectrum of B is given by the interpolation series

$$u(k) = \sum_\zeta \frac{\theta\Pi(k)}{k-\zeta}\,\frac{u(\zeta)}{dS/dk|_\zeta}$$

For similar calculations in the general case of nonunitary S see ref. 10.

Using the functional model, we reduce the spectral properties of a dissipative operator to studying the analytic properties of a single (scalar or matrix) analytic function $S(k)$. In particular, the distribution and the number of eigenvalues are defined by uniqueness theorems which describe the properties of zero-sets of an analytic function (as a function of its smoothness in the closed upper half-plane). For instance, it was shown [4] that the number of eigenvalues of the Schrödinger operator with the complex potential $q(x)$ satisfying the property

$$|q(x)| \leq C\exp(-d|x|^\alpha), \qquad d > 0$$

is finite if $\alpha \geq 1/2$, but may be infinite if $\alpha < 1/2$. In the latter case the set of all accumulation points of the eigenvalues is situated on the positive half-axis of the spectral parameter and it is represented generally by a fractal set of measure zero which has a positive Hausdorf dimension μ, where $\mu \leq (1 - 2\alpha)/1 - \alpha)$.

The systems of eigenvectors of the discrete spectrum of a dissipative operator are not generally orthogonal, but sometimes they can be connected by an invertible transformation to some orthogonal and normalized basis.

Such systems are called *Riesz bases*. For Riesz bases the equation representing the infinite-dimensional Pythagorean theorem, i.e., the Parseval identity, is replaced by a corresponding bilateral inequality which can be used to prove the convergence of spectral decompositions. The system of eigenfunctions of a dissipative operator forms a Riesz base if and only if for any uniformly bounded interpolation data defined on the zero set of the scattering matrix the bounded solution of the interpolation problem exists in $U^j_- K$. In the scalar case this is the well-known Carleson condition

$$\prod_{l \neq m} \left| \frac{k_l - \bar{k}_m}{k_l - k_m} \right| \geq \delta > 0$$

The connection between spectral decompositions and interpolation problems is an important fact of harmonic analysis; see, for instance, ref. 11 and the bibliography therein. In particular, in the $U^j_- K$-spectral representation of the dilation the biorthogonal decomposition in the eigenfunction becomes an interpolation series. The important question of separating the spectral subspaces N_d, N_s, N_c corresponding to the discrete, singular, and (absolutely) continuous spectrum is reduced to the question of embedding theorems for some classes of analytic functions [12]. The convergence and summability of spectral decompositions over the absolutely continuous spectrum of a dissipative operator depends on the distribution of real zeros of the corresponding scattering matrix—so-called spectral singularities. It is extremely important that for any dissipative operator there exist a "canonical" system of the eigenfunctions of absolutely continous spectrum [10], which is formed by orthogonal projections onto K of the eigenfunctions of the part of the self-adjoint dilation in the invariant subspace orthogonal to the reducing subspace H_- generated by the incoming channel.

$$H_- = \bigvee_0^{\infty} e^{iL't} D_-.$$

The role of the spectral density for this system is played by $S^{-1}(k) - S(k)$ [10].

Returning to the classical problem of exponential bases on a finite interval $(0, 2\pi)$, one can formulate a test for a base property in terms of the generating function f of the exponentials $e^{ik_l x}$, which is an entire function of exponential type 2π vanishing at the points k_l. Assuming that f is bounded in the upper half-plane and all zeros of f are situated in upper half-plane, we can reduce the problem to the question of joint completeness of the systems of eigenfunctions of two mutually adjoint operators B, B^+,

$$\langle Bu, v \rangle \equiv \langle u, B^+ v \rangle$$

the first of them having the characteristic function $S(k) = f(k)[\overline{f(\bar{k})}]^{-1}$. The answer to the question of the base property of exponentials on the finite interval is the following [14]:

Theorem 1. The system of exponentials $\{e^{ik_l x}\}$ forms a Riesz base in $L_2(0, 2\pi)$ if only if the following conditions are fulfilled:

$$1.\ \inf_m \prod_{l \neq m} \left| \frac{k_l - k_m}{\overline{k_l - k_m}} \right| \geq \delta > 0$$

$$2.\ \sup \frac{1}{\Delta} \int_\Delta |f|^2\, dx \cdot \frac{1}{\Delta} \int_\Delta |f|^{-2}\, dx < \infty, \quad \Delta \subset R$$

Condition 2 is the celebrated Muckenhoupt condition [15], which is actually equivalent to the geometric Kato zero-index condition for the orthogonal projections P_1, P_2 onto the pair of subspaces $N_1 = fH^2_+$ and $N_2 = L_2 \ominus fH^2_-$ in L_2:

$$\|P_1 - P_2\| < 1$$

which is equivalent to the invertibility of operators $P_1P_2P_1$, $P_2P_1P_2$ in L_1, L_2, respectively. Note that the exponentials on a finite interval are eigenfunctions of a nonstandard spectral problem for the momentum operator $i\, d/dx$ with the “spread” boundary condition, which cannot be analyzed by standard tools of spectral theory. However, using our geometric method of orthogonal projections, we can reduce the situation to a problem of classical operator theory.

There exists a large class of nonstandard problems of spectral analysis [10] which were mentioned in the beginnning of this paper. These are spectral problems for operator functions which arise when reducing a nunitary dynamical group to some observation subspace K. Only rare cases exist when the reduced dynamics is described by a contracting semigroup, but we always can associate a generalized spectral problem with it by considering the corresponding Fourier transform—the compressed resolvent—the *Livshic matrix* [5]:

$$\int_0^\infty e^{-ikt} P_K U_t P_K\, dt = iP_K[L - kI]^{-1} P_K, \qquad \text{Im}\, k \leq 0$$

The operator function in K which arises in this way is initially an analytic function in the lower half-plane, but in many interesting cases it possesses an analytic continuation to the upper half-plane with some singularities—so-called resonances. The analysis of resonances is important for the description of the asymptotic behavior of the restricted dynamical group, and we need a sort of spectral decomposition associated with resonance states, appearing in the form of residues of the analytic continuation of the compressed resolvent at the resonances, similar to the eigenfunctions which constitute the residues of the resolvent of the self-adjoint operator at the isolated eigenvalues. Unfortunately the result of the analytic continuation of the compressed resolvent is not generally a resolvent of any operator. The most important property of resolvents—the Hilbert identity—is not generally valid for compressed resolvents. Nevertheless it can be analyzed by our methods in some interesting cases using the Kato zero-index condition [18]. One can formulate the following general principle for analytical continuation of the compressed resolvent of operators which generally are not generators of Lax–Phillips groups:

Let us assume that for given self-adjoint operator A there exist a unitary group U_t commuting with A and possessing orthogonal incoming and outgoing subspaces $D_\pm$ and a coinvariant subspace K. If the operator A is a rational function of generators of the group U_t, then the compressed resolvent $P_K(A-\lambda I)^{-1}\big|_K$ possesses an analytical continuation onto the nonphysical sheet of the spectral parameter λ across the continuous spectrum of A as a linear combination of resolvents of generators of reduced semigroups $P_K U_t\big|^K$, $P_K U_t^+\big|^K$ with coefficients which are polynomials of generators.

This principle was observed first—for an operator with two-band spectrum—in ref. 16 and proved in ref. 18 for perturbed Jacobian matrices. This prinicple enables us to apply the method of the functional model to the spectral analysis of Livshic matrices of operators with band spectrum. In particular it enables us to investigate the completeness and basis property of the system of residual vectors which correspond to the poles of the Livshic matrix on the nonphysical sheet of the spectral parameter. The last problem can be reduced to the problem of *joint completeness* of the eigenvectors of generators of the reduced dynamics and the adjoint operators and the *joint* base property. This gives new results on completeness and expansion by resonances for important quantum and acoustic problems, in particular for periodic problems. But proper spectral analysis of resonances for periodic or quasiperiodic Schrödinger operators cannot be done using the standard Lax–Phillips and Sz-Nagy–Foias techniques, which are applicable only to simple domains, but requires development of the corresponding technical tools for multiconnected domains on the complex plane [16, 17]. The investigation of the corresponding few-dimensional problems is an attractive field of research for analysts and mathematical physicists.

4. AVERAGED DYNAMICS ON MARKOV BACKGROUND. DILATION AND SPECTRAL ANALYSIS

Returning to our initial motivation, we mention here one more effect which destroys irreversibility. This is the stochastic background which is practically present in every physical experiment. Initially it may arise from the quantum nature of our universe. The importance of the presence of the stochastic background has been pointed out repeatedly by Prigogine in his books on nonequilibrium statistical mechanics (see, for instance, ref. 19). One can show that the presence of the stochastic background can be treated technically in a way very similar to the case of reduced dynamics [20].

Consider a quantum Hamiltonian $L(x)$ which depends on some Markov process $x(t)$ on a discrete finite stochastic space X. We assume that the transition probabilities of the process are governed by the Fokker–Planck equation for transition probabilities, with some symmetric positive stochastic matrix M.

$$\frac{dp}{dt} + Mp = 0$$

Let us assume that the Hamiltonian H depends on time via some Markov process $x(t)$ taking values in the space X of stochastic states. The natural description of the corresponding evolution is given by the Liouville equation for the density matrix $\rho(t)$, which is realized as an element of the space Q of Hilbert–Schmidt-type operators supplied with the inner product $\langle \rho_1, \rho_2 \rangle = \mathrm{Trace}\rho_2^{+}\rho_1$:

$$\frac{1}{i}\frac{d\rho}{dt} = H\rho - \rho H \equiv H^{\times}\rho$$

The transformation of the initial data of the last equation is produced by the unitary T-product $U(t)$: $\rho(0) \to \rho(t)$, which is generated by the superoperator $H^{\times}$ as a solution of the following differential equation:

$$\frac{1}{i}\frac{dU}{dt} = H^{\times}U$$

$$U(0) = I$$

The evolution operators $U(t)$ along the fixed trajectory of the Markov process obviously do not form a group or a semigroup. Nevertheless, averaging $U(t)$ over all trajectories with fixed ends

$$x(0) = x_0, \qquad x(t) = x_t$$

yields a *contracting* semigroup generated by a *dissipative operator*. In the case of a discrete stochastic space the spectrum of the corresponding generator B may have several one-dimensional branches in the upper half-plane of the spectral parameter which are isomorphic to the absolutely continuous spectrum of the time-independent Liouvillian with the stochastic process "frozen."

The following statement summarizes some results concerning the quantum evolution on a Markov background (see, for instance, ref. 23).

Theorem 2. Assume that the stochastic space X is finite and that the evolution of the transition probabilities of $x(t)$ is governed by the Fokker–Planck differential equation for transition probabilities in X with a positive matrix $M{:}E \to E$, dim E = card $X = d$:

$$\frac{dP}{dt} + MP = 0$$

Then the generator of the quantum evolution averaged over all trajectories with fixed ends is a dissipative operator in quantum-stoachastic space $Q \times E$ given by the following construction:

$$L_M = \{\text{diag}_X H^{\times}(x)\} + iM \times I_Q$$

Here I_Q is the unit operator in the quantum space Q of density matrices, and $H^{\times}\rho \equiv H\rho - \rho H$ for any $\rho \in Q$.

Stronger results including scattering on Brownian particle have been given, e.g., by Cheremshantsev [24]. The described construction is similar to the algebraic operation of braiding groups. We call it *braiding evolutions* $e^{iH(x)t}$ *via the Markov Process* $x(t)$. The result of braiding is the evolution generated by the operator L_M.

The explicit construction suggested above of the self-adjoint dilation for Schrödinger operators with complex potentials is quite general and can be applied to the generators of the averaged dynamics. These dissipative operators are strong perturbations of self-adjoint operators and they usually have branches of continuous spectrum in the upper half-plane.[3]

In the simplest case of a two-point stochastic state the dissipative generator of the averaged dynamics is represented in matrix form:

$$\begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} + i\frac{\kappa^2}{2}\begin{pmatrix} I_q & -I_q \\ -I_q & I_q \end{pmatrix}$$

where $L_\pm$ are quantum Liouvillians corresponding to the stochastic states $\pm$, and I_q is the unit operator in the quantum space of density matrices. In this case the Fokker–Plank operator and the second term of the generator of the averaged dynamics coincide up to a factor κ^2 with the orthogonal projection P_a in the quantum-stochastic space onto the subspace E_a of elements which

[3] Sometimes they are weak perturbations of normal operators [20].

are antisymmetric with respect to the stochastic variables. Due to the idempotency of the projections $[(P_a)^2 = P_a]$ this generator has the form

$$A + iG^2$$

where $G = \kappa P_a$.

The dilation of this operator is constructed by attaching to the quantum-stochastic space two orthogonal "incoming" and "outgoing" channels $D_\pm$

$$D_+ = E \times L_2(0, \infty), \qquad D_- = E \times L_2(-\infty, 0)$$

with momentum operators

$$2i \frac{du_\pm}{d\xi}, \qquad \xi \in (0, \pm\infty)$$

acting on $D_\pm$. In our case the perturbation iG^2 is not a relatively weak operator with respect to the real part A. The construction of the minimal self-adjoint dilation is described by the following general assertion [20]:

Theorem 3. Consider a dissipative operator L represented in the form of a sum of a self-adjoint operator A acting on a Hilbert space K and an imaginary part iG^2 constructed as a square of a bounded nonnegative operator G. We assume that Range $G = E$:

$$L = A + iG^2$$

Then the operator which is defined on the orthogonal sum of the space K and the incoming and outgoing channels $D_\pm = E \times L_2(0, \pm\infty)$ by the formula

$$\mathcal{L}\begin{pmatrix} u_- \\ u \\ u_+ \end{pmatrix} = \begin{pmatrix} 2i \dfrac{du_-}{d\xi} \\ Au + iG[u_-(0) + u_+(0)] \\ 2i \dfrac{du_+}{d\xi} \end{pmatrix}$$

with the boundary condition

$$[u_+(0) - u_-(0)] = Gu$$

is the minimal self-adjoint dilation of L. The absolutely continuous spectrum of it fills the real axis (generally with varying multiplicity). If the generalized limits of the resolvent of the real part A on some dense linear subspace E', $E' \subset E$ exist, then the eigenfunctions of the dilation can be represented through these limits.

In particular, this operator has two orthogonal systems $\psi_\pm$ of eigenfunctions of the scattered-wave type, which form a basis of (reducing) invariant subspaces $H_\pm$ generated by $D_\pm$, respectively, and two orthogonal systems of eigenfunctions of absolutely continuous spectrum $\psi^>$, $\psi^<$ in complementary subspaces $H^>$, $H^<$.For instance,

$$\psi_- = \begin{pmatrix} e^{-(i/2)\lambda\xi}\nu \\ u \\ e^{-(i/2)\lambda\xi}S^+(\lambda)\nu \end{pmatrix}, \nu \in E'$$

The transmission coefficient S^+ is nontrivial only in the subspace E and is represented there by the formula

$$S^+(\lambda) = \frac{I_E - iGR^A(\lambda - i0)G}{I_E + iGR^A(\lambda - i0)G}$$

through the generalized limit $R^A(\lambda - i0)$ of the resolvent of the real part A, and the central component u of ψ_- is given by the formula

$$u = -iR^A(\lambda - i0)G[S^+\nu + \nu]$$

The eigenfunctions $\psi^<$ of the complementary component of $\mathcal{L}$ in $H \ominus H_-$ vanish on D_- and are represented through the eigenfunctions of the operator L (which play the role of the central component) and exponentials in the outgoing space:

$$\psi^< = \begin{pmatrix} 0 \\ u \\ Gue^{-(i/2)\lambda\xi} \end{pmatrix}$$

Here u is the properly normed eigenfunction of the operator L:

$$Au + iG^2u = \lambda u$$

A similar construction and the corresponding fact remain true in the general case of a Markov process generated by some Dirichlet form. The new subject of nonequilibrium statistical mechanics—the S matrix, which appears as a scattering matrix of the self-adjoint dilation of B—may play an important role in the description of the large-time asymptotic behavior of quantum systems approaching equilibrium and/or exhibing chaotic behavior.

ACKNOWLEDGMENTS

The research of the author was supported by various grants, including Grant 3368152 from the Marsden Fund of the Royal Society of New Zealand and grants from the Russian Academy, in particular Grant RFFI 97-01-01149.

REFERENCES

[1] J. B. Fourier, *Théorie Analytique de la Chaleur*, Paris (1822).
[2] D. Hilbert, Wesen and Ziele einer Analysis der wnendlich vielen unabhangige Variablen, *Rend. Circ. Mat. Palermo* **27**, 59–74 (1909).
[3] H. Spohn, The spectrum of the Liouville–von Neumann operator, *J. Math. Phys.* **17**, 57–60 (1976).
[4] B. Pavlov, Spectral theory of nonselfadjoint differential operators, in *Proceedings of the International Congress of Mathemeticians, August 16–24, Warszawa* (1983), Vol. 2, pp. 1011–1025.
[5] M. S. Livshic, *Operators Oscillations Waves*, Nauka, Moscow (1966).
[6] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert space*, North-Holland, Amsterdam (1970).
[7] P. Lax, and R. Phillips, *Scattering Theory*, Academic Press, New York (1967).
[8] I. E. Antoniou, Intrinsic irreversibility of quantum systems and rigged Hilbert space for the Lees–Friedrichs model, in *Proceedings of 2nd Wigner Symposium, July 15–20* (1991).
[9] V. Adamjan, and D. Arov, On unitary couplings of semi-unitary operators, *Mathem. Issled.* **1**(2), 3–64 (1966). [English transl., *Am. Math. Soc. Transl. 2,* **95**(1970).]
[10] B. Pavlov, Spectral Analysis of a dissipative singular Schrödinger operator in terms of a functional model, in *Partial Differential Equations*, M. Shubin, ed., Springer (1995), pp. 87–153.
[11] N. Nikolski and B. Pavlov, Eigenvector bases of completely nonunitary contractions and the characteristic function, *Izv. Acad. Nauk SSSR Ser. Mat.* **34**, 90–133 [English transl., *Math. USSR Izv.* **4** (1970)].
[12] B. Pavlov, On separation conditions of a dissipative operator, *Izv. Akad. Nauk USSR Ser. Mat.* **39**, 123–148 (1975) [English transl., *Math. USSR Izv.* **9** (1975)].
[13] B. Pavlov, Selfadjoint dilation of the Schrödinger operator and its resolution in terms of eigenfunctions, *Mat. Sbornik* **102**(144), 4 (1977) [English transl., *Math. USSR Sbornik* **31**, 4 (1977)].
[14] B. Pavlov, The base property of a system of exponentials and Muchenhoupt condition, *Dokl. Akad. Nauk USSR* **247**, 37–40 (1979) [English transl., *Sov. Math. Doklady* **20**, 4 (1979)].
[15] R. Hunt, B. Muckenhoupt, and R. L. Wheeden, Weighted norm inequalities for conjugate function and Hilbert transform, *Trans. Am. Math. Soci.* **176**, 227–251 (1973).
[16] B. Pavlov and S. Fedorov, The group of shifts and harmonic analysis on a Riemann surface genus one, *Alg. Anal.* **1**(2), 132–168 (1989) [English transl., *Leningrad Math. J.* **1**, (1989)].
[17] S. Fedorov, On harmonic analysis in multiply connected domain and character-automor phic Hardy spaces, *Spb. Math. J.* **9**(2), 192–240 (1997).
[18] B. Pavlov, Zero index of pair of projectors and expansion by resonance states for operators with band spectrum, Preprint ESI **139**, Vienna (1994).
[19] I. Prigogine and I. Stengers, *Entre le temps et l'eternité*, Paris, Fayard (1988).
[20] B. Pavlov. Quantum dynamics on Markov background and irreversibility, in *Nonlinear Dynamics, Chaotic and Complex Systems*, I. Infeld, R. Zelazny, and A. Galkowsky, eds. Cambridge University Press, Cambridge, pp. 198–205.
[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, New York (1979), Vols. 1–5.
[22] B. Pavlov, Harmonic analysis on a Riemannn surface and nonphysical sheet for the band spectrum, in *Functional Analysis and Related Topics*, S. Koshi, ed., World Scientific, Singapore (1991).

[23] R. Griego and R. Hersh, Theory of random evolution with applications to partial differential equations, *Trans. Am. Math. Soc.* **156**, 405–418.
[24] S. Cheremshantsev, Theory of scattering by Brownian particle, *Trudy Matem. Inst. Steklov* **184** (1990) [English transl. *Proc. Steklov Inst. Math., Acad. Sci. USSR*, Issue 2 (1991)].
[25] P. Kurasov and B. Pavlov, Nonphysical sheet and Schrödinger evolution, in *Proceedings of International Workshop Mathematical Aspects of Scattering Theory and Applications. St. Petersburg, May 1991*, pp. 72–81.